ISRAEL JOURNAL OF MATHEMATICS, Vol. 39, Nos. 1-2, 1981

THE CONSISTENCY OF Ext(G, Z) = Q

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ABSTRACT

For abelian groups, if $V = L$, Ext(G, Z) cannot have cardinality $\aleph_0$. We show that G.C.H. does not imply this. See Hiller and Shelah [2], Hiller, Huber and Shelah [3], Nunke [5] and Shelah [6, 7, 8] for related results. We use the method of [7].

THEOREM 1. *Suppose the universe V satisfies* G.C.H., *and K is a divisible countable (abelian) group (i.e. $|K| \leqq \aleph_0$). Then for some forcing notion P, in V^P for some abelian group G,* Ext(G, Z) = K.

COROLLARY 2. *It is consistent that for some group G,* Ext(G, Z) = Q (Q — *the rationals as an additive group,* Z — *the integers as an additive group).*

REMARK. (1) This answers questions from Hiller and Shelah [2], Huber, Hiller and Shelah [3] and Nunke [5]; remember that if $V = L$, Ext(G, Z) $\neq \aleph_0$. The result was announced in [9].

(2) The group we get is $\aleph_1$-free and of power $\aleph_1$.

(3) Instead of "K countable", we can demand "$|K| \leqq \aleph_2$"; the proof will not change significantly. We can change $|G|$ and $|K|$, but we have not checked carefully.

(4) It is well known that Ext(G, Z) is divisible.

PROOF. Let K be the direct sum of $K_{\mathbf{p}}$ ($\mathbf{p}$ a prime natural number or zero), where K_0 is torsion free and for $\mathbf{p} \neq 0$ $(\forall x \in K_{\mathbf{p}})$ $(\exists n)$ $\mathbf{p}^n x = 0$. This is possible as K is divisible, and each $K_{\mathbf{p}}$ is divisible. Let $K^1_{\mathbf{p}} = \{x \in K_{\mathbf{p}} : \mathbf{p}x = 0\}$, so $K^1_{\mathbf{p}}$ is a vector space over $\mathbf{Z}/\mathbf{pZ}$. So let $B_{\mathbf{p}} \subseteq K_{\mathbf{p}}$ be a basis of $K^1_{\mathbf{p}}$ (as a vector space over the rationals for $\mathbf{p} = 0$, and over $\mathbf{Z}/\mathbf{pZ}$ otherwise). Let $B = \bigcup_{\mathbf{p}} B_{\mathbf{p}}$, so as K is countable, $|B| \leqq \aleph_0$. Let $K^1_{\mathbf{p}} \subseteq K_{\mathbf{p}}$ $[K^1 \subseteq K]$ be the subgroup generated by $B_{\mathbf{p}}[B]$ (for $\mathbf{p} > 0$ this is not new).

[†]The author would like to thank the United States–Israel Binational Science Foundation for partially supporting this research by Grant No. 1110.

Received February 12, 1980

Choose S to be a stationary costationary set of limit ordinals $<\omega_1$. We shall define a group G of the following form: G is freely generated by x_i $(i<\omega_1)$ and $z_{\delta,n}$ $(\delta\in S,\ n<\omega)$, with the only identities

$$\mathbf{p}(\delta,n)z_{\delta,n}=x_\delta-\tau^\delta_n \qquad \text{where } \tau^\delta_n=\sum_{\zeta\in a(\delta,n)}x_\zeta$$

where $\mathbf{p}(\delta,n)$ is a strictly increasing function of n (for each δ separable); for $\delta\in S$, η_δ is an increasing ω-sequence of *successor* ordinals converging to δ, $a(\delta,n)=\{\eta_\delta(l):\ l\leqq k_\delta(n),\ l>k_\delta(m)$ for every $m<n\}$.

NOTATION. If h is a function from ω_1 to $\mathbf{Z}$, $\tau=\sum_{i=0}^n c_ix_{l(i)}$ $(c_i\in\mathbf{Z})$ then $h(\tau)=\sum c_ih(l(i))$. We let $\mathbf{p}$ be a prime number.

FACT A. $\mathrm{Ext}(G,\mathbf{Z})$ is isomorphic to E_0/E_1, where E_0 is the set of functions from $S\times\omega$ into $\mathbf{Z}$, addition is defined coordinatewise; E_1 is the subgroup of $f\in E_0$ such that for some $h:\omega_1\to\mathbf{Z}$, $f\approx\hat h$, i.e., $f(\delta,n)=\hat h(\delta,n)\bmod \mathbf{p}(\delta,n)$ (for every $(\delta,n)\in S\times\omega$), where $\hat h$ is defined by $\hat h(\delta,n)=h(\delta)-h(\tau^\delta_n)$.

PROOF OF FACT A. Like that of [10] 3.3.

Now we shall define the group G by defining the $a(\delta,n)$ and an embedding of B into E_0/E_1; we do it by forcing, to simplify the proof.

An element q of $P_1=Q_0$ is a triple:

$$a(\delta)^q=\langle\langle a(\delta,n)^q,\mathbf{p}(\delta,n)^q\rangle:n<\omega\rangle \quad\text{for}\quad \delta\in S,\quad \delta<\delta_0,$$

$$f^q_s\ (s\in B),\qquad h^q_s\ (s\in B-B_0),$$

such that the $\langle a(\delta,n):n<\omega\rangle$ are as mentioned above: $a(\delta,n)$ is a non-empty finite subset of δ, $\max a(\delta,n)<\min a(\delta,n+1)$, $\delta=\sup\{\min a(\delta,n):n<\omega\}$, f^q_s is a function from $(S\cap\delta_0)\times\omega$ into $\mathbf{Z}$, and for $s\in B_{\mathbf{p}}$, $\mathbf{p}\neq0$, $\mathbf{p}f_s\approx\hat h_s$ (where $(\mathbf{p}f_s)(i)=\mathbf{p}(f_s(i))$) and $h_s:\delta_0\to\mathbf{Z}$.

Also $\mathbf{p}(\delta,n)$ is a prime natural number, $\mathbf{p}(\delta,n)<\mathbf{p}(\delta,n+1)$. The order is natural.

Clearly there is a P_0-name $\underset{\sim}{G}$ defined by $\underset{\sim}{a}(\delta,n)$, $\underset{\sim}{p}(\delta,n)$ and $\underset{\sim}{f}_s$ $(s\in B)$, and let $\underset{\sim}{f}_t=\sum\underset{\sim}{f}_{s_i}$ where $t=\sum s_i$, $s_i\in B$ (i.e. $t\in K^1$). Clearly $\underset{\sim}{f}_t\in E_0$.

Clearly in V^{Q_0}, $\mathrm{Ext}(\underset{\sim}{G},\mathbf{Z})$ is too big. So we define an iterated forcing P_i $(i\leqq\omega_2)$, with countable support, $P_{i+1}=P_i*\underset{\sim}{Q}_i$ such that for each $i>0$, $\underset{\sim}{Q}_i$ "kills" an undesirable member of $\mathrm{Ext}(\underset{\sim}{G},\mathbf{Z})$. More elaborately, for each i, $\underset{\sim}{f}_i$ is a P_i-name of a member of E_0, such that for some $\mathbf{p}(i)$ either $\mathbf{p}(i)=0$, and $\phi\Vdash^{P_i}$ "$(\forall n>0)\ (\forall t\in K^1)\ (n\underset{\sim}{f}_i-\underset{\sim}{f}_t\notin E_1)$", *or* $\mathbf{p}=\mathbf{p}(i)$ is prime >0 and

$$\phi \Vdash^{P_i} \text{“}\mathbf{p} f_i \in E_0 \wedge (\forall n)(\forall t \in K^1_p)(0 < n < \mathbf{p}) \rightarrow [n\underset{\sim}{f_i} - \underset{\sim}{f_t} \notin E_0]\text{”}$$

Now in V^{P_i}, $Q_i = \{h : \text{for some } \alpha,\ h : \alpha \rightarrow \mathbf{Z}$, and for every $\delta \leqq \alpha$, $\delta \in S$, $f_i(\delta, n) = h(\delta) - h(\tau^\delta_n) \bmod \mathbf{p}(\delta, n)$ (if $\delta = \alpha$ this means there is such $h(\delta)$)$\}$. The order: inclusion.

FACT B. (1) (in V^{P_i}) If $h \in Q_i$, $\operatorname{Dom} h = \alpha$, $\alpha \leqq \beta$, then there is h^*, $h \leqq h^* \in Q_i$, $\operatorname{Dom} h^* = \beta$. Moreover if h' is a finite function from $[\alpha, \beta)$ to $\mathbf{Z}$ we can demand $h' \subseteq h^*$, except when $\alpha \in S \cap \operatorname{Dom} h'$.

(2) (in V^{P_i}) $\phi \Vdash^{Q_i}$ “$\underset{\sim}{f_i} \in E_1$”.

PROOF. (1) By induction on β. For $\beta \notin S$, totally trivial; for $\beta \in S$, we first define $h^* \restriction \{\eta_\beta(n) : n < \omega,\ \alpha \leqq \eta_\beta(n)\}$ appropriately, and then define $h^* \restriction \eta_\beta(n)$ by induction on n.

(2) Follows from (1).

So $P_i = \{p : \operatorname{Dom} p$ is a countable subset of i, $p(j)$ a P_j-name of a member of Q_j for $j \in \operatorname{Dom} p$, i.e., $\phi \Vdash^{P_j}$ “$p(j) \in Q_j$”$\}$. (We shall write $p(i)(\xi) = c$ for $p \restriction i \Vdash^{P_i}$ “$p(i)(\xi) = c$”.) The order is $p_1 \leqq p_2$ if $i \in \operatorname{Dom} p_1$ implies $p_2 \restriction i \Vdash^{P_i}$ “$p_1(i) \leqq p_2(i)$”. As in [7]:

FACT C. (1) For every $p \in P_i$ there is $p' \in P_i$, $p \leqq p'$, and for some δ, $\forall \alpha \in \operatorname{Dom} p$, $\operatorname{Dom} p'_i(\alpha) = \delta$ and $p'(\alpha) \in V$. Such p' is called of height δ.

(2) If p_n has height α_n, $p_n \leqq p_{n+1}$, $\alpha_n < \alpha_{n+1}$, $\bigcup_{n<\omega} \alpha_n = \delta \notin S$ then $\bigcup_{n<\omega} p_n \in P$.

(3) P_{ω_2} satisfies the $\aleph_2$-c.c. and does not add new ω-sequences. By suitable bookkeeping we can assume every P_{ω_2}-name $\underset{\sim}{f}$ of a function as above is $\underset{\sim}{f}$ for some i.

(4) If in the forcing by P_{ω_2}, it is forced that, for every t and p, “$t \in K^1_{\mathbf{p}} \Rightarrow f_t \notin E_1$” *then* $\operatorname{Ext}(\underset{\sim}{G}, \mathbf{Z}) \simeq K$ (note that E_1 depends on the universe we are dealing with).

PROOF. As in [7], (1), (2), (3) hold. Let us prove (4). Remember that by Fact A, $\operatorname{Ext}(\underset{\sim}{G}, \mathbf{Z}) \simeq E_0/E_1$. As, e.g., by Fuchs [1], E_0/E_1 is a divisible group; it is enough to check that:

(a) $t \in K^1$, $t \neq 0$ implies $f_t \notin E_1$,

(b) for $f \in E_0 - E_1$, for some $n > 0$, and $t \in K^1$, $nf \notin E_1$, and $nf - f_t \in E_1$.

Now (a) follows immediately by the hypothesis whereas (b) follows by Fact C3 (and the definition of Q_i).

So the rest of the proof is dedicated to the proof that the hypothesis of Fact C4 holds. So suppose $t_* \in K^1_{p_*}$, $t_* \neq 0$, $\underset{\sim}{h}$ a P_{ω_2}-name, $q_* \in P_{\omega_2}$,

$$(*) \qquad q_* \Vdash^{P_{\omega_2}} \text{“} \underset{\sim}{f}_{t_*} \approx \hat{\underset{\sim}{h}} \text{”}.$$

As P_{ω_2} satisfies the $\aleph_2$-chain condition, we can replace P_{ω_2} by P_i, $(i < \omega_2)$ and choose a minimal such i (i.e., i minimal such that there are a P_i-name $\underset{\sim}{h}$ and $q_* \in P_i$, so that $q_* \Vdash^{P_i}$ “$\underset{\sim}{f}_{t_*} \approx \hat{\underset{\sim}{h}}$”). Those i, t_*, p_*, q_*, $\underset{\sim}{h}$ are fixed for the rest of the proof.

Before we prove we note some easy facts on the forcings.

FACT D. (1) If $\alpha < \beta$, $p \in P_\alpha$, $q \in P_\beta$, $(q \restriction \alpha) \leqq p$, then $r = p \vee q$ is their least upper bound (where $\operatorname{Dom} r = \operatorname{Dom} p \cup \operatorname{Dom} q$, $r(j)$ is $p(j)$ for $j \in \operatorname{Dom} p$ and $q(j)$ for $j \in \operatorname{Dom} q - \operatorname{Dom} p$).

(2) If $p \in P_\alpha$, $\alpha_0 < \cdots < \alpha_{n-1} < \alpha$, h_l a finite function from ω_1 to $\mathbf{Z}$ for $l < n$ such that $p \restriction \alpha_l \Vdash^{P_{\alpha_l}}$ “$\operatorname{Dom}(p(\alpha_l)) < \min \operatorname{Dom} h_l$” then there is $q, p \leqq q \in P_\alpha$, such that for $l < n$, $q \restriction \alpha_l \Vdash^{P_{\alpha_l}}$ “$h_l \subseteq q(\alpha_l)$”.

PROOF. (1) See [7]; easy to check.

(2) Prove by induction on α_{n-1}, using Fact B1.

FACT E. If $q \in P_i$, $\alpha_0 < \cdots < \alpha_{n-1} < i$, $\bar{\alpha} = \langle \alpha_0, \cdots, \alpha_{n-1} \rangle$ then for some q', $q \leqq q' \in P_i$, q' has height and for every q'', $q' \leqq q'' \in P_i$, $\operatorname{Pos}_{\bar{\alpha}}(q') = \operatorname{Pos}_{\bar{\alpha}}(q'')$ where $\operatorname{Pos}_{\bar{\alpha}}(q^0) = \{\langle c^0, \cdots, c^{2m-1} \rangle$: for every $\zeta_0 < \omega_1$ for some successor ζ, $\zeta_0 < \zeta < \omega_1$ and $r_0, \cdots, r_{m-1} \in P_i$, $q^0 \leqq r_0, \cdots, q^0 \leqq r_{m-1}$, $r_0 \restriction \alpha_{n-1} = r_1 \restriction \alpha_{n-1} = \cdots = r_{m-1} \restriction \alpha_{n-1}$, and $r_l(\alpha_{n-1})(\zeta) = c^{2l}$ (for $l < m$) and $r_l \Vdash^{P_i}$ “$\underset{\sim}{h}(\zeta) = c^{2l+1}$” for $l < m\}$. Note that $\alpha_0, \cdots, \alpha_{n-2}$ were not used, so $\operatorname{Pos}_{\bar{\alpha}}(q^0)$ depend only on q, α_{n-1}, and $\operatorname{Pos}_{\bar{\alpha}}(q^0)$ decrease when α_{n-1}, q^0 increase.

PROOF. Easy by Fact C2.

So w.l.o.g.

ASSUMPTION E1. (1) Either (α) or (β) where

(α) i is a successor (ordinal) or of cofinality $\aleph_0$, and for arbitrarily large $\alpha < i$, $\operatorname{Pos}_{\langle \alpha \rangle}(q_*) = \operatorname{Pos}_{\langle \alpha \rangle}(q')$ for $q' \in P_i$, $q' \geqq q_*$;

(β) i has cofinality $\aleph_1$, and there is $\alpha_* < i$ such that $\operatorname{Pos}_{\langle \alpha_* \rangle}(q_*) = \operatorname{Pos}_{\langle \alpha \rangle}(q')$ whenever $\alpha_* \leqq \alpha < i$, $q_* \leqq q' \in q_*$.

(2) Also q_* has height γ^*.

NOTATION. An $\bar{\alpha}$ whose last element is among the α's in (α) if (α) holds and is $\geqq \alpha_*$ if (β) holds, is called good.

DEFINITION F. We call a *candidate* a sequence $\bar{u} = \langle\langle a_n, \mathbf{p}_n\rangle : n < n_*\rangle$ such that a_n is a finite non-empty subset of successor ordinals $< \omega_1$, $\max a_m < \min a_{m+1}$ for $m < n_*$, $\mathbf{p}_m$ prime (so $\bar{u}^i = \langle\langle a^i_n, \mathbf{p}^i_n\rangle : n < n^i_*\rangle$ etc.).

For a good $\bar{\alpha} = \langle \alpha_0, \cdots, \alpha_{m-1}\rangle$, $0 \leqq \alpha_0 < \cdots < \alpha_{m-1} < i$, $\alpha_0 = 0 \Rightarrow \mathbf{p}_* \neq 0$, and g, $g : \operatorname{Range} \bar{\alpha} \to \omega$ let

$$T(g, \bar{\alpha}, \bar{u}) = \{t : t \text{ a function from } \{\langle \alpha_l, k\rangle : l < m, g(\alpha_l) \leqq k < n_*\},$$
$$t(\alpha_l, k) \in \{c \in \mathbf{Z} : 0 \leqq c < \mathbf{p}_k\}\}.$$

We call $\bar{q} = \{q_t : t \in T\}$ an $(g, \bar{\alpha}, \bar{u})$-tree, if $T = T(g, \bar{\alpha}, \bar{u})$, $q_* \leqq q_t$ (n_* — from $\bar{u}$) and if $t \in T$, $l < l(\bar{\alpha})$, $g(\alpha_l) \leqq k < n_*$ then

(a) $t(\alpha_l, k) = q_t(\alpha_l)(\tau_k) \bmod \mathbf{p}_k$ where $\tau_k = \sum_{\zeta \in a_k} x_\zeta$ and $\alpha_l > 0$,

(b) if $t_1 \restriction (\alpha_l \times \omega) = t_2 \restriction (\alpha_l \times \omega)$ then $q_{t_1} \restriction \alpha_l = q_{t_2} \restriction \alpha_l$,

(c) if $\alpha_0 = 0$, then

$$t(\alpha_0, k) = \sum_{s \in S} n_s h^{q_t}_s(\tau_k) \bmod \mathbf{p}_k \quad \text{where } t_* = \sum_{s \in S} n_s t_s \quad (t_s \text{ is from } B_{\mathbf{p}_*}).$$

FACT G. Suppose $g, \bar{\alpha}, \bar{u}, \bar{q}$ are as in Definition F. Then we can find a_{n_*}, $\mathbf{p}_{n_*}$, c_*, $\bar{q}^1$ such that (it seems $c_* = 0$ always)

(a) $\bar{q}^1$ is a $(g, \bar{\alpha}, \bar{u}^1)$-tree,

(b) $\bar{u}^1 = \bar{u}^\frown\langle a_{n_*}, \mathbf{p}_{n_*}\rangle$,

(c) if $t_1 \in T(g, \bar{\alpha}, \bar{u}^1)$, $t \in T(g, \bar{\alpha}, \bar{u})$ and $t \subseteq t_1$ then $q_t \leqq q^1_{t_1}$,

(d) for every $t_1 \in T(g, \bar{\alpha}, \bar{u}^1)$, $q_{t_1} \Vdash$ "$\underset{\sim}{h}(\tau_{n_*}) \neq c_* \bmod \mathbf{p}_{n_*}$".

We delay the proof of Fact G, but first we prove from it the desired contradiction.

Let $\underset{\sim}{h}, q_* \in N \prec (H(\aleph_2), \in, P, \Vdash)$, N countable, $\delta^* = N \cap \omega_1 \in S$. We define by induction on n, g^n, $\bar{\alpha}^n$, $\bar{u}^n$, $\bar{q}^n$ such that

(a) $\bar{q}^n$ is a $(g^n, \bar{\alpha}^n, \bar{u}^n)$-tree,

(b) g^n, $\bar{\alpha}^n$, $\bar{u}^n \in N$, $\bar{\alpha}^n$ good,

(c) $q_* \leqq q^0_t$ for every $t \in T(\bar{g}^0, \bar{\alpha}^0, \bar{u}^0)$,

(d) $g^n \subseteq g^{n+1}$, $\operatorname{Range} \bar{\alpha}^n \subseteq \operatorname{Range} \bar{\alpha}^{n+1}$, $\bar{u}^{n+1} \restriction n = \bar{u}^n$, $\bar{u}^n$ has length n,

(e) if $t \in T(\bar{g}^n, \bar{\alpha}^n, \bar{u}^n)$, $t^*_1 \in (\bar{g}^{n+1}, \bar{\alpha}^{n+1}, \bar{u}^{n+1})$, $t \subseteq t^*_1$ then $q^n_t \subseteq q^{n+1}_{t_1}$,

(f) $\delta^* = \bigcup_{n<\omega} \delta_n$, $\delta_n < \delta_{n+1} < \delta^*$ and $t \in T(g^{n+1}, \bar{\alpha}^{n+1}, \bar{u}^{n+1})$ implies q^{n+1}_t is bigger than some condition of height β^n_t, $\delta_n \leqq \beta^n_t$ and every $\zeta \in N \cap i$ belongs to $\bigcup_{n<\omega} \operatorname{Range} \bar{\alpha}^n$ except 0 when $\mathbf{p}_* = 0$,

(g) for every $n < \omega$ for some c^n_*, $c^n_* \in \mathbf{Z}$, $0 \leqq c^n_* < \mathbf{p}^{n+1}_n$, and for every $t \in T(\bar{g}^{n+1}, \bar{\alpha}^{n+1}, \bar{u}^{n+1})$, $q^{n+1}_t \Vdash^{P_i}$ "$\underset{\sim}{h}(\tau_n) \neq c^n_*$ mod $\mathbf{p}^{n+1}_n$".

The definition is possible by Fact G (plus a trivial work). We concentrate on the case $\mathbf{p}_* = 0$.

Clearly there are $q^n \in Q_0$ such that for every $t \in T(\bar{g}^n, \bar{\alpha}^n, \bar{u}^n)$, $q^n_t(0) = q^n$. Now clearly $q^\omega = \bigcup q^n \in Q_0$; and as in [7] 1.7, 1.8, for every $q', q^\omega \leqq q' \in P_0$, if $q' \Vdash$ "$a(\delta^*, n) = a^{n+1}_n$, $\mathbf{p}(\delta^*, n) = p^{n+1}_n$ for $n < \omega$" there is r, $q' \leqq r \in P_i$, $q_* \leqq r$, and for every n for some $t \in T(\bar{g}^n, \bar{\alpha}^n, \bar{u}^n)$, $q^n_t \leqq r$.

So r forces that

(i) for every n, $\underset{\sim}{h}(\tau_n) \neq c^n_*$ mod $\mathbf{p}^{n+1}_n$,

(ii) suppose $t_* = \Sigma_s n_s t_s$ $(t_s \in B_{\mathbf{p}_*})$ then as $q_* \Vdash$ "$f_{t_*} \approx \underset{\sim}{h}$", $q_* \leqq r$, clearly $\Sigma_s r_s \underset{\sim}{f}_{t_s}(\delta^*, n) = \underset{\sim}{h}(\delta^*) - \underset{\sim}{h}(\tau_n)$ mod $\mathbf{p}^{n+1}_n$.

Notice that when choosing q' we have total freedom to choose the $f_{t_m}(\delta^*, n) \in \mathbf{Z}$. So for each $c \in \mathbf{Z}$, for some n we can contradict the possibility $\underset{\sim}{h}(\delta^*) = c$. There is no problem to complete the definition of $f_t(\delta^*, n)$ $(t \in B)$, $h_t(\delta, n)$ $(t \in \bigcup_{p \neq 0} B_p)$ to get q'.

For $\mathbf{p}_* \neq 0$, the problem is that $h_t \restriction \bigcup_{l<\omega} a(\delta^*, l) = h_t \restriction \text{Range}(\eta_{\delta^*})$ in fact determine $f_t \restriction \{(\delta^*, n) : n < \omega\}$, for $t \in B_{\mathbf{p}_*}$; however, the definition of the tree provides us with enough freedom for the choice of $h_{t_*}(\eta_\delta(l))$, i.e., we choose $h_s(\delta)$. Let us enumerate $\mathbf{Z} : \mathbf{Z} = \{d_n : n < \omega\}$ and choose $h_s(\tau_n)$ $(s \in S)$ (where $t_* = \Sigma_{s \in S} n_s t)$ such that $\Sigma_{s \in S} n_s h_s(\delta) - d_n - \Sigma_{s \in S} n_s h_s(\tau_n) = c^*_n$ mod $\mathbf{p}(\delta, n)$.

So we are left with:

Proof of Fact G. Let $T = T(g, \bar{\alpha}, \bar{u})$. It is easy to see that

Fact H. If $\bar{q}^0 = \langle q^0_t : t \in T \rangle$ is a $(g, \bar{\alpha}, \bar{u})$-tree, $t_0 \in T$, $q^0_{t_0} \leqq q'_{t_0} \in P_i$, then we can find q'_t $(t \in T - \{t_0\})$ such that $q_t \leqq q'_t$ and $\langle q'_t : t \in T \rangle$ is a $(g, \bar{\alpha}, \bar{u})$-tree.

Now the following fact is crucial.

Fact I. One of the following cases holds:

(a) there are $c(l)$ $(l = 0, 1, 2)$ in $\mathbf{Z}$ such that $c(1) \neq c(2)$ and $\langle c(0), c(1), c(0), c(2) \rangle \in \text{Pos}_{\bar{\alpha}}(q_*)$,

(b) there are $c(l)$ $(l = 0, 1, 2, 3, 4, 5)$ such that $\langle c(l) : l < 6 \rangle \in \text{Pos}_{\bar{\alpha}}(q_*)$, but $c(2l) \mapsto c(2l+1)$ is not a linear function, i.e., there are no rational numbers d_1, d_2 such that $c(2l+1) = d_1 c(2l) + d_2$,

(c) there are $c(l)$ $(l < 8)$ such that $\langle c(l) : l < 4 \rangle \in \text{Pos}_{\bar{\alpha}}(q_*) \langle c(l) : 4 \leqq l < 8 \rangle \in \text{Pos}_{\bar{\alpha}}(q_*)$ but $(c(3) - c(1))/(c(2) - c(0)) \neq (c(7) - c(5))/(c(6) - c(4))$ (both well defined).

PROOF OF FACT I. Let $\gamma = \alpha_{n_*-1}$ and $\underset{\sim}{h}_\gamma$ be the P_γ-name of $\bigcup\{q(\gamma): q$ is in the generic set$\}$. So if (a) fails, then for some P_γ-name $\underset{\sim}{F}$

$$q_* \Vdash^{P_i} \text{“}\underset{\sim}{h}(\zeta) = \underset{\sim}{F}(\zeta, \underset{\sim}{h}_\gamma(\zeta))\text{”} \qquad \text{for every successor } \zeta \geqq \gamma^*, \quad \zeta < \omega_1$$

(so $\underset{\sim}{F}$ is a function from $\omega_1 \times \mathbf{Z}$ to $\mathbf{Z}$). If also (b) fails then there are P_γ-names $\underset{\sim}{d}_1$, $\underset{\sim}{d}_2$ (of functions from ω_1 to $\mathbf{Z}$) such that

$$(q_* \restriction \gamma) \Vdash^{P_i} \text{“}\underset{\sim}{F}(\zeta, c) = \underset{\sim}{d}_1(\zeta)c + \underset{\sim}{d}_2(\zeta) \text{ for every successor } \zeta < \omega_1, \zeta \geqq \gamma^*\text{”}.$$

If also (c) fails then $d_1(\zeta) = d_1 \in \mathbf{Z}$ for some d_1.

So suppose (a), (b) and (c) fail, and let $G_i \subseteq P_i$ be generic, $q_* \in G_i$. Then in $V[G_i]$, $f_\gamma \approx \hat{h}_\gamma$, $f_i \approx \hat{h}$. Let $h^* = h - d_1 h_\gamma$, then $f_i - d_1 f_\gamma \approx \hat{h}^*$. Now $f_i, f_\gamma \in V[G_\gamma]$ (where $G_\gamma = G_i \cap P_\gamma$) so if we prove $h^* \in V[G_\gamma]$ we shall get a contradiction (to the requirement on $\underset{\sim}{f}_\gamma$ in the definition of our iterated forcing). Now for $\zeta \geqq \gamma^*$ successor, $h^*(\zeta) = \tilde{d}_2(\zeta)$, and the function d_2 belongs to $V[G_\gamma]$. So $h^* \restriction \{\zeta + 1: \zeta \geqq \gamma^*\} \in V[G_\gamma]$. Also all our forcings do not add reals, hence $h^* \restriction \gamma^* \in V[G_\gamma]$. So $h^* \restriction \{\zeta < \omega_1 : \zeta$ non limit$\} \in V[G_\gamma]$, but we can construct $h^* \restriction \{\delta < \omega_1 : \delta$ limit$\}$ from f_i, f_γ, $h^* \restriction \{\zeta < \omega_1 : \zeta$ non limit$\}$, by the equations

$$f_i(\delta, n) - d_1 f_\gamma(\delta, n) = h^*(\delta) - \sum_{\zeta \in a(\delta, n)} h^*(\zeta) \bmod \mathbf{p}(\delta, n)$$

as all elements of $a(\delta, n)$ are successor ordinals. So we finish the proof of Fact I.

CONTINUATION OF THE PROOF OF FACT G. Now we choose a prime natural number $\mathbf{p}_{n_*} > \mathbf{p}_{n_*-1}$ such that $c(2) - c(1) \neq 0 \bmod \mathbf{p}_{n_*}$ if (a) holds and $(c(3) - c(1))/(c(2) - c(0)) \neq (c(5) - c(1))/(c(4) - c(0)) \bmod \mathbf{p}_{n_*}$ (so $c(2) - c(0) \neq 0 \bmod \mathbf{p}_{n_*}$) if (b) holds, and $(c(3) - c(1))/(c(2) - c(0)) \neq (c(7) - c(5))/(c(6) - c(4)) \bmod \mathbf{p}_{n_*}$ if (c) holds (and so that divisions are not by zero).

So now $T^1 = T(g, \bar{\alpha}, \bar{u}^\wedge\langle\langle a_{n_*}, p_{n_*}\rangle\rangle)$ is defined, though a_{n_*} is still not defined. Let for a finite set a of successor ordinals $< \omega_1$ but $> \operatorname{Max} a_{n_*-1}$ (a will be an initial segment of the a_{n_*} we shall construct)

$$R_a = \{\bar{r} : \bar{r} = \langle r_t : t \in T^1\rangle \quad \text{a} \quad (\bar{g}, \bar{\alpha}, \bar{u}^\wedge\langle\langle a, p_{n_*}\rangle\rangle)\text{-tree}$$
$$\text{and } t_0 \in T, t \in T^1, t_0 \subseteq t \text{ implies } q_{t_0} \subseteq r_t \text{ and } r_t \text{ determine } \underset{\sim}{h}(\zeta) \text{ for each } \zeta \in a\}.$$

It is easy to check that $R_a \neq \emptyset$, and that as T^1 is finite it suffices to prove (for proving Fact G, thus finishing the poof)

FACT J. If $\bar{r}^0 \in R_a$, $t_1 \in T^1$, then we can find a_1, $a \subseteq a_1$, $\operatorname{Max} a < \operatorname{Min}(a_1 - a)$, or $a_1 - a = \emptyset$ and $\bar{r}' \in R_{a_1}$ such that:

(1) for every $t \in T^1$, $r_t^0 \leqq r_t^1$,

(2) $r'_{t_1} \Vdash^{P_1}$ "$\Sigma_{\zeta \in a_1} h(\zeta) \neq 0 \bmod \mathbf{p}_{n_*}$",

(3) for $t \in T^1$, $t \neq t_1$ $r^1_t \Vdash$ "$\Sigma_{\zeta \in a_1 - a} \underset{\sim}{h}(\zeta) = 0 \bmod \mathbf{p}_{n_*}$".

PROOF OF FACT J. If $r^0_{t_1} \Vdash$ "$\Sigma_{\zeta \in a} \underset{\sim}{h}(\zeta) \neq 0 \bmod \mathbf{p}_{n_*}$" we can let $a_1 = a$. So assume this fails.

On R_a there is a natural order $\bar{r}^2 \leqq \bar{r}^3$ iff $r^2_t \leqq r^3_t$ for every $t \in T^1$. As in Fact C it is easy to show that above every $\bar{r} \in R_a$ there is some $\bar{r}'$ of height α for some α (i.e., each r'_t ($t \in T^1$) has height α). Now we can define

$$\begin{aligned}\mathrm{Pos}^a(\bar{r}) = \{\langle c^t_l : t \in T^1, l \leqq l(\bar{\alpha})\rangle : &\text{for every } \zeta_0 < \omega_1 \text{ for some successor } \zeta,\\ &\zeta_0 < \zeta < \omega_1 \text{ there is } \bar{r}^1 \in R_a,\ \bar{r} \leqq r^1 \text{ and } r^1_t(\alpha_l)(\zeta) = c^t_l\\ &\text{for } l < l(\bar{\alpha}) \text{ and } r^1_t \Vdash \text{“}\underset{\sim}{h}(\zeta) = c^t_{l(\bar{\alpha})}\text{”}\}.\end{aligned}$$

As in the proof of Fact E, w.l.o.g. our $\bar{r}^0$ is such that $\mathrm{Pos}^a(\bar{r}^0) = \mathrm{Pos}^a(\bar{r})$ for any $\bar{r}$, $\bar{r}^0 \leqq \bar{r} \in R_a$. Now we should consult Fact I, i.e., which of the three possibilities there holds. Note that we shall add many times ($\mathbf{p}_{n_*} - 1$) instead of subtracting.

First assume that (a) holds and $c(l)$ ($l < 3$) exemplifies it. By Fact H, there are $\langle c^t_l : t \in T^1, l \leqq l(\bar{\alpha})\rangle$, $\langle d^t_l : t \in T^1, l \leqq l(\bar{\alpha})\rangle$ in $\mathrm{Pos}^a(\bar{r})$ such that $c^t_l = d^t_l$ except for $t = t_1$, $l = l(\bar{\alpha})$, and $c^{t_1}_{l(\bar{\alpha})} = c(1)$, $d^{t_1}_{l(\bar{\alpha})} = c(2)$; remember that in constructing a tree the interactions are only up to $\alpha_{n_*} - 1$. So we can find $\zeta^m < \omega_1$, $\bar{r}^m \in R_a$ by induction on $m \leqq \mathbf{p}_{n_*}$ such that:

(i) $\bar{r}^m \leqq \bar{r}^m \leqq r^1$, $\max(a) < \zeta^m < \zeta^{m+1}$, ζ^m a successor,

(ii) for every $t \in T^1$, and $l < l(\bar{\alpha})$ and $m > 0$,

$$r^{m+1}_t(\alpha_l)(\zeta^m) = c^t_l, \qquad r^1_t(\alpha_l)(\zeta^0) = d^t_l,$$

(iii) for every $t \in T^1$

$$r^{m+1}_t \Vdash \text{“}\underset{\sim}{h}(\zeta^m) = c^m_{l(\bar{\alpha})}\text{”}, \qquad r^1_t \Vdash \text{“}\underset{\sim}{h}(\zeta^1) = d^t_{l(\bar{\alpha})}\text{”}.$$

So $\bar{r}^{\mathbf{p}}$, $\{\zeta_l : l < \mathbf{p}\} \cup a$ (where $\mathbf{p} = \mathbf{p}_{n_*}$) are as required.

So we turn to case (b) and let $c(l)$ ($l = 0,1,2,3,4,5$) exemplify this. We can find k_l ($l < 3$) such that $\Sigma_{l<3} k_l c(2l) = 0 \bmod \mathbf{p}_{n_*}$; $\Sigma_{l<3} k_l = 0 \bmod \mathbf{p}_{n_*}$ but $\Sigma_{l<3} k_l c(2l+1) \neq 0 \bmod \mathbf{p}_{n_*}$ w.l.o.g. $k_l > 0$, let $k = \Sigma_{l<3} k_l$.

It is easy to see that we can find $\langle c^{t,m}_l : t \in T^1, l \leqq l(\bar{\alpha})\rangle \in R_a$, for $m = 0,1,2$, such that $c^{t,m}_l = c^{t,0}_l$ for $t \neq t_1$ or $l \leqq l(\bar{\alpha}) - 2$, and $c^{t_1,m}_{l(\alpha)-1} = c(2m)$, $c^{t_1,m}_{l(\bar{\alpha})} = c(2m+1)$.

Now we can define $\bar{r}^l$, ζ^l, $m(l)$ ($1 \leqq l \leqq k$) by induction on l such that ($\bar{r}^0$ is given) $\bar{r}^l \leqq \bar{r}^{l+1}$, $\operatorname{Max} a < \zeta^1$, $\zeta^l < \zeta^{l+1}$, $m(1) = \cdots = m(k_0) = 0$, $m(k_0+1) = \cdots = m(k_0+k_1) = 1$, $m(k_0+k_1+1) = \cdots = m(k_0+k_1+k_2) = 2, \cdots, r^l_t(\alpha_l)(\zeta^l) = c^{t,m(l)}_l$, $r^l_t \Vdash$ "$\underset{\sim}{h}(\zeta^l) = c^{t,m(l)}_l$".

Clearly the last $\bar{r}^l$, $\bar{r}^k$ is the $\bar{r}'$ required in the Fact.

For the case (c) holds, the proof is similar.

References

1. L. Fuchs, *Infinite Abelian Groups*, Academic Press, 1970, 1972.

2. H. L. Hiller and S. Shelah, *Singular cohomology in L*, Israel J. Math. **26** (1977), 313–319.

3. H. L. Hiller, M. Huber and S. Shelah. *The structure of* Ext(G, $\mathbf{Z}$) *and* $V = L$, Math. Z. **162** (1978), 39–50.

4. T. Jech, *Set Theory*, Academic Press, 1978.

5. R. K. Nunke, *Whitehead Problem*, Proc. Second New Mexico Abelian Group Symp. on Abelian Group Theory, Springer Verlag Lecture Notes **616**, pp. 240–250.

6. S. Shelah, *Infinite abelian groups. Whitehead problem and some constructions*, Israel J. Math. **18** (1974), 243–256.

7. S. Shelah, *Whitehead groups may not be free even assuming CH*, Israel J. Math. **28** (1977), 193–203.

8. S. Shelah, *On uncountable abelian groups*, Israel J. Math. **32** (1979), 311–330.

9. S. Shelah, *On well ordering and more on Whitehead problem*, Notices Amer. Math. Soc. **26** (1979), A-442.

10. S. Shelah, *Whitehead groups may not be free even assuming CH, II*, Israel J. Math. **35** (1980), 257–285.

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